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A PROBLEM IN ANALYTIC GEOMETRY WITH A MORAL

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ALTHOUGH the problem which I propose to treat is of a very elementary character, the result is not without elegance. My object in publishing this paper is less to make a modest contribution to geometrical knowledge (in fact it seems quite incredible that the results here obtained should be new) than to use this problem to point a moral which I feel sure is much needed by many mathematical students, and perhaps by some who no longer regard themselves as students. Anyone who reads this paper will doubtless feel sure that he would have been able to solve the problem correctly himself, as nothing but the most elementary methods are employed. If, as I strongly advise him to do, he tries to solve the problem without looking at §§2, 3 of the present paper, he may be of another opinion. It is only when used in this way that the paper can be of any real service.

1. The problem and the method of solving it. The problem we wish to treat is the determination of all non-degenerate conics which intersect at right angles a given real equilateral hyperbola at all points (real or imaginary) where they meet it. We will take as coordinate axes the principal axes of the equilateral hyperbola, and as unit of length its semi transverse axis. The equation of the hyperbola is then :

$$x^2 - y^2 - 1 = 0. \tag{1}$$

Consider a second conic

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0, \tag{2}$$

which is ultimately to be so determined that it cuts (1) at right angles, but which for the moment we regard as arbitrarily given.

Let us first determine the locus of a point P whose polars with regard to (1) and (2) are perpendicular. If (x_1, y_1) are the coordinates of P, the polars in question are

$$x_1x - y_1y - 1 = 0, \tag{3}$$

$$(Ax_1 + By_1 + D)x + (Bx_1 + Cy_1 + E)y + (Dx_1 + Ey_1 + F) = 0; \tag{4}$$

and the necessary and sufficient condition that (3) and (4) be perpendicular is:

$$(Ax_1 + By_1 + D)x_1 - (Bx_1 + Cy_1 + E)y_1 = 0.$$

Dropping the subscripts and rearranging the terms we see that the desired locus is the conic:

$$Ax^2 - Cy^2 + Dx - Ey = 0. \tag{5}$$

Now the necessary and sufficient condition that (2) cut (1) at right angles is that the locus (5) pass through all the points of intersection of (1) and (2); or in other words, that the conics (1), (2), (5) be members of a pencil, *i. e.* that it be possible to express one of them in the form $u + \lambda\, v = 0$, where $u = 0$ and $v = 0$ are the equations of the other two.

It is at this point that the reader is advised to lay aside this paper and work out the problem for himself.

2. The solution. It is clear that if (2) cuts (1) at right angles, they cannot coincide, and therefore they determine a pencil. Denoting the first members of the equations (1), (2), (5) by u, v, w respectively, the necessary and sufficient condition that (5) belong to this pencil is that two constants α, β exist such that $\alpha u + \beta v \equiv w$. Equating the coefficients of like terms on the two sides of this identity, we are led to the following equations:

$$\begin{aligned} \alpha + A\beta &= A, \\ 2B\beta &= 0, \\ -\alpha + C\beta &= -C, \\ 2D\beta &= D, \\ 2E\beta &= -E, \\ -\alpha + F\beta &= 0. \end{aligned} \tag{6}$$

We must determine all sets of values (not all zero) of A, B, C, D, E, F which permit of the determination of two constants α, β such that the equations (6) are satisfied.

The second of these equations tells us that either $\beta = 0$ or $B = 0$. These two possibilities must be considered separately.

I. $\beta = 0$. Here the sixth equation (6) gives $\alpha = 0$, and the remaining equations give

$$A = C = D = E = 0.$$

The equation (2) then has the form:

$$2Bxy + F = 0,$$

and, since this is to be a non-degenerate conic, $B \neq 0$, and we may write this equation in the form:

$$xy = \lambda. \tag{7}$$

This is a first family of conics which all cut (1) at right angles. They are the equilateral hyperbolas whose axes are the asymptotes of (1).

II. $\beta \neq 0$. The fourth and fifth equations (6) may be written:

$$D(2\beta - 1) = 0, \qquad E(2\beta + 1) = 0,$$

and we distinguish three cases:

$$\text{II}(a): \ \beta = \tfrac{1}{2}; \quad \text{II}(b): \ \beta = -\tfrac{1}{2}; \quad \text{II}(c): \ \beta \neq \pm\tfrac{1}{2}.$$

We will consider these cases in succession.

II(a). $\beta = \frac{1}{2}$. The second and fifth equations (6) now give:

$$B = E = 0.$$

The first, third, and sixth equations may be written:

$$A = 2a, \ C = \tfrac{2}{3}a, \ F = 2a.$$

Accordingly (2) may be written:

$$2a(x^2 + \tfrac{1}{3}y^2 + 1) + 2Dx = 0.$$

Since this is to be a non-degenerate conic, $a \neq 0$, and we may write this equation in the form:

$$3x^2 + y^2 + \lambda x + 3 = 0. \tag{8}$$

This is a second family of conics which cut (1) at right angles. They are all the ellipses of eccentricity $\sqrt{2/3}$ which pass through the two imaginary points $(0, \pm\sqrt{-3})$ and whose minor axes are parallel to the transverse axis of (1).

II(b). $\beta = -\frac{1}{2}$. This case is quite like the previous one, and gives as a third family of conics which cut (1) at right angles:

$$x^2 + 3y^2 + \lambda y - 3 = 0. \tag{9}$$

These are all the ellipses of eccentricity $\sqrt{2/3}$ which pass through the points $(\pm\sqrt{3}, 0)$ and whose major axes are parallel to the transverse axis of (1).

II(c). $\beta \neq 0$, $\beta \neq \pm\frac{1}{2}$. Here we have

$$B = D = E = 0,$$

while from the first, third, and sixth of equations (6) we find, by letting $\lambda = -1/\beta$, that equation (2) takes the form

$$(10) \qquad \frac{x^2}{\lambda + 1} + \frac{y^2}{\lambda - 1} = 1,$$

a fourth family of conics, from which however the conic $\lambda = 0$ must be excluded,* which cut (1) at right angles. These are all the conics (ellipses and hyperbolas) which are confocal with (1) and different from it.

Thus we have finally the result:

The necessary and sufficient condition that a non-degenerate conic cut (1) *at right angles at all points of intersection is that it belong to one of the one parameter families* (7), (8), (9), (10).

3. The Moral. The lesson to be drawn from this problem, as the reader who has worked it through for himself has probably already noticed, is that unknown constants may have the value zero and that it is unsafe to divide by them until we have assured ourselves that this is not the case. Thus at the very start the fact that the three conics (1), (2), (5) form a pencil is expressed most generally by writing

$$pu + qv + rw \equiv 0$$

where p, q, r are constants not all three zero. By dividing through by p, q, r respectively this may be put in any one of the following forms:

$$\beta_1 v + \gamma_1 w \equiv u,$$
$$\alpha_2 u + \gamma_2 w \equiv v,$$
$$\alpha u + \beta v \equiv w.$$

The last of these is the form we used. The first two, which seem at first sight equally general, will not do for our purposes since p and q are both zero in the case we have numbered I. If we had started from either of these first two forms we should therefore have obtained only a part of the solution of our problem; and it should be noticed that what we should lose (viz. case I) is by no means a "special case," but is a case of exactly the same degree of generality as the cases II(a), II(b), II(c) which we should obtain.

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* Strictly speaking the two conics $\lambda = \pm 2$ must also be omitted since we have assumed $\beta \neq \pm \frac{1}{2}$. These two curves, however, do cut (1) at right angles since they are included in the families (8) and (9) respectively.

The same remarks apply to the discussion of equations (6). We must not, for instance, overlook the possibility that $2\beta + 1 = 0$, nor must we suppose that if we do overlook it we are merely omitting a "special," that is, a more or less exceptional case.

4. Generalizations. The method here explained can be applied to the general problem of determining all conics which cut any given conic at right angles, and in space, of determining all quadric surfaces which cut a given quadric surface at right angles. In such general cases the difficulties, or rather the necessity for care, become much greater; but no essentially new point is involved. A systematic working through of this problem, including a geometrical discussion of the families of curves and surfaces obtained, would serve as an excellent subject for an undergraduate thesis for an ambitious student, provided the temptation of publication could be withstood.

An easy and interesting special case is the determination of all conics which cut a parabola, say $y^2 = 4x$, at right angles. It will be found that there are three one parameter families of such conics.